A PROPOSED GENERALIZED CONSTITUTIVE EQUATION FOR NONLINEAR PARA-ISOTROPIC MATERIALS*

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INTRODUCTION

With the advent of finite element models of varying complexity the focus of solutions to problems in solid mechanics has shifted very strongly into the direction of more accurate material description. This is especially true for materials for which strength characteristics vary widely with state of stress. In particular, concrete which is non-isotropic at any level of deformation and is also non-linear in terms of stress-strain relationships has been singled out for intensive study. This includes work on constitutive relations (refs. 1 to 5), and failure (refs. 3,4, and 6 to 15).

Related developments in failure theories are those included in references 16 to 25.

This list is by no means exhaustive but formed a background basis which led to the model proposed in this paper. In particular, the developments in the areas of maximum deformation theory and the Von Mises-Hencky theory provided motivation for the concept used here (ref. 26).

FAILURE SURFACE

The proposed generalized constitutive equation is an extension of the work of Hu and Swartz (26) on the study of the failure of materials so that for any kind of material in any state of stress its mechanical behavior can be characterized by a single functional. According to the theory, a material failure initiates when the state of stress at a point is such that the following functional reaches a threshold value F_o:

$$F(\vec{\sigma}) = \frac{\alpha J_2}{2[\sigma_f^c \sigma_f^t - (\sigma_f^c - \sigma_f^t) J_1]} + (1-\alpha) \frac{\left[\frac{\sigma_1}{E(\sigma_1)} - \mu\left(\frac{\sigma_2}{E(\sigma_2)} + \frac{\sigma_3}{E(\sigma_3)}\right)\right](\sigma_f^c + \sigma_f^t)}{\varepsilon_f^t(\sigma_f^c + J_1) + \mu\varepsilon_f^c(\sigma_f^t - J_1)} \tag{1}$$

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In this equation

α = a material related scalar factor which is determined by experimental data.

$J_1 = \sigma_1 + \sigma_2 + \sigma_3$.

$J_2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2$.

$\sigma_1, \sigma_2, \sigma_3$ = the components of principal stresses, tensile stress has positive value. A vector of principal stresses is expressed by $\vec{\sigma}$.

σ_f^t, σ_f^c = the absolute value of the ultimate stresses of the material under uniaxial tension and compression, respectively.

$\varepsilon_f^t, \varepsilon_f^c$ = the absolute values of strain component in the direction of uniaxial force when stress reaches the corresponding ultimate value.

$E(\sigma)$ = the secant modulus of elasticity of the material subjected to uniaxial stress (tension or compression as appropriate).

μ = the Poisson's ratio of the material (related to stress level).

CONSTITUTIVE EQUATIONS

Non-linear response will take place when the increment of the functional is in an increasing manner and its value is beyond some threshold level (possibly zero). The material will be fractured at points where the state of stress reaches the surface of failure, i.e., $F(\vec{\sigma}) = 1$.

Between the initial state to fracture, the response is assumed to be characterized by Drucker's (17) theorem of orthogonality with the use of the functional proposed by the generalized failure theory, eqn. (1). That is, if the increment of principal strain components is decomposed into linear and non-linear increments,

$$\{d\varepsilon\} = \{d\varepsilon^e\} + \{d\varepsilon^p\}, \tag{2}$$

the non-linear part is characterized by

$$\{d\varepsilon^p\} = G(\vec{\sigma})\{g(\vec{\sigma})\}\ dF. \tag{3}$$

where

$G(\vec{\sigma})$ = a scalar function of stresses,

$\{g(\vec{\sigma})\}$ = a unit vector of the gradient of the functional at the point of interest.

Note that G values (under uniaxial loading) can be determined by some stress-

strain relationship. In the general case, especially for those materials like glass or concrete, the G function in uniaxial tension differs from that of uniaxial compression remarkably. Therefore a weighted average is proposed. According to this, the relationship between the vectors and principal strain increment and the principal stress increment is proposed to be characterized by the following generalized constitutive equation:

$$\{d\varepsilon\} = \left[\frac{1}{E} \begin{Bmatrix} 1 & -\mu & -\mu \\ -\mu & 1 & -\mu \\ -\mu & -\mu & 1 \end{Bmatrix} + G(\vec{\sigma}) \ \{g(\vec{\sigma})\} \ \lfloor \nabla F(\vec{\sigma}) \rfloor \right] \{d\sigma\} \tag{4}$$

In this E = the modulus of elasticity of the material, $\lfloor \nabla F \rfloor$ = the row vector of the gradient of the functional at the point of interest, and the function $G(\vec{\sigma})$ is calculated by

$$G(\vec{\sigma}) = \sqrt{\frac{\left[\sum_{i=1}^{3} (G_i(\sigma_i)\sigma_i)^2\right]}{\sum_{i=1}^{3} \sigma_i^{\,2}}} \ H(F-F_o) \ H(\Delta F) \tag{5}$$

which is a weighted average according to the values of the corresponding principal stress components. $G_i(\sigma_i)$ depends on the value of the i-th uniaxial principal stress and is calculated according to

$$G_i(\sigma_i) = G^+[F(\sigma_i)] \ H(\sigma_i) + G^-[F(\sigma_i)] \ H(-\sigma_i). \tag{6}$$

In this formula, H is a unit step function. Let $\Delta\varepsilon_i^p = \varepsilon_i^p(\sigma_i + \Delta\sigma_i) - \varepsilon^p(\sigma_i)$, $\Delta F = F(\sigma_i + \Delta\sigma_i) - F(\sigma_i)$.
Then,

$$G^+[F(\sigma_i)] = \lim_{\Delta F \to 0} \frac{1}{\Delta F} \left| \frac{\Delta\varepsilon_i^p}{g_i} \right| , \quad \sigma_i \geq 0, \tag{7a}$$

$$G^-[F(\sigma_i)] = \lim_{\Delta F \to 0} \frac{1}{\Delta F} \left| \frac{\Delta\varepsilon_i^p}{g_i} \right| , \quad \sigma_i \leq 0, \tag{7b}$$

where g_i is the i-th component of the unit vector {g}.

If $\sigma_f^{\,c} = \sigma_f^{\,t}$, $\varepsilon_f^{\,t} = \varepsilon_f^{\,c}$ and $\alpha = 1$ is selected, the proposed failure theory agrees with the Von Mises' theory and the generalized constitutive equation reduces to the well known Prandtl-Reuss stress-strain relation (refs. 4,11).

NUMERICAL EXAMPLE

Applying the proposed theory to plain concrete for the purpose of illustration equ. (1) is modified to read

$$F = \frac{\alpha}{2}\,\frac{J_2}{\sigma_f^{\,c}\sigma_f^{\,t} - (\sigma_f^{\,c}-\sigma_f^{\,t})J_1} + (1-\alpha)\frac{[\sigma_1+\mu(\sigma_2+\sigma_3)](\sigma_f^{\,t}+\sigma_f^{\,c})}{\sigma_f^{\,t}\sigma_f^{\,c}(1+\mu) + (\sigma_f^{\,t}-\mu\sigma_f^{\,c})J_1} \qquad (8)$$

The implication used is an invariant modulus of elasticity and for concrete, so-called initial yield occurs at about .45 f_c' .

Using test data for concrete (3) the level surfaces of the functional, the stress-strain curves of uniaxial tension and compression tests and the variations of $G^+[F(\sigma_i)]$ and $G^-[F(\sigma_i)]$ are shown in Figs. 1-3. The best value of α for concrete is 0.46 which has been used in these curves.

Note in Fig. 1 the shape of the curve in the tension-compression zone follows very closely the shape of the experimental curve obtained by Kupfer, Hilsdorf and Rusch (3). The stress-strain curves presented in Figs. 2 and 3 were obtained using testing equipment described in Ref. 27. Using these data, Equations 8 for F and 7a and 7b for G^+ and G^- were evaluated numerically to obtain the curves displayed.

CONCLUSIONS

A proposed constitutive model for non-linear materials has been presented. The primary virtues of the model are its logical combination of distortion states, inherent simplicity and generality.

Results presented for the model applied to concrete show good agreement with published experimental data for failure. The model can be readily incorporated into existing computer codes provided sufficient experimental supportive data are available.

As reported elsewhere (10), for instance, the experimentally obtained parameter α for cast iron is around 0.93.

The proposed constitive equation is presently being utilized in the development of a finite element code for determination of unstable crack growth and stress intensity in concrete beams.

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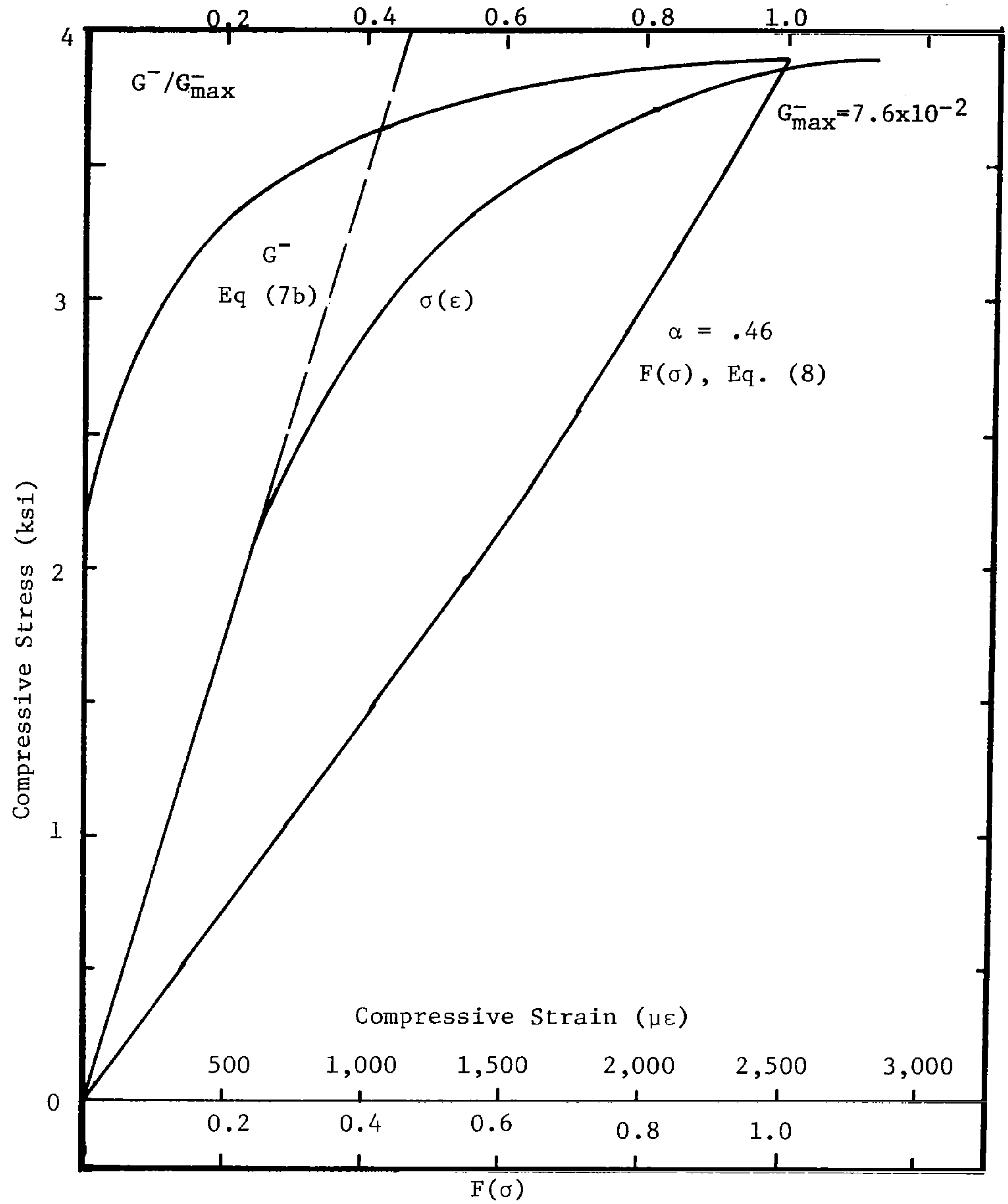

Figure 3.- Data of uniaxial compressive test of concrete.

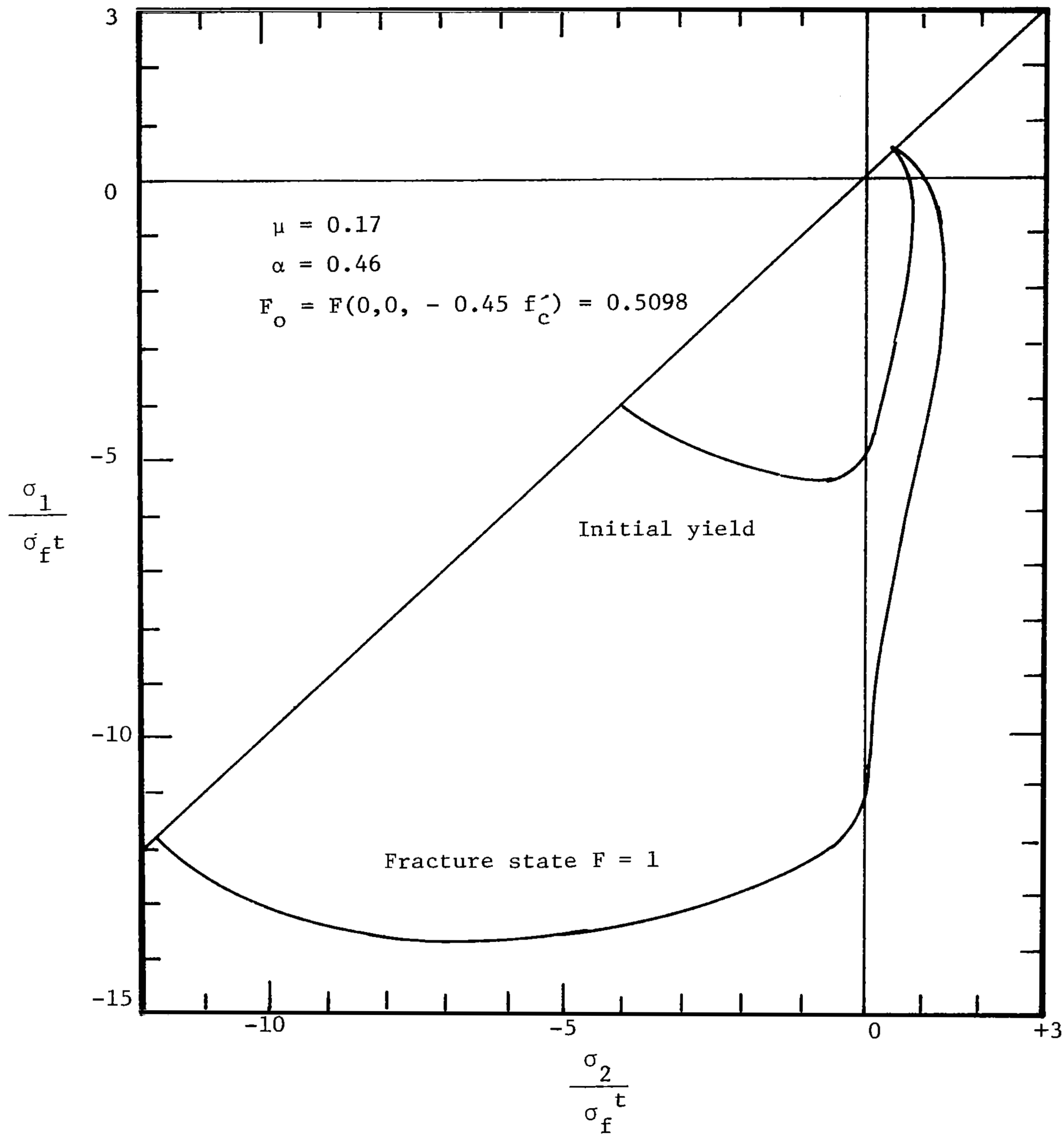

Figure 1.- Yield and fracture surfaces of concrete.

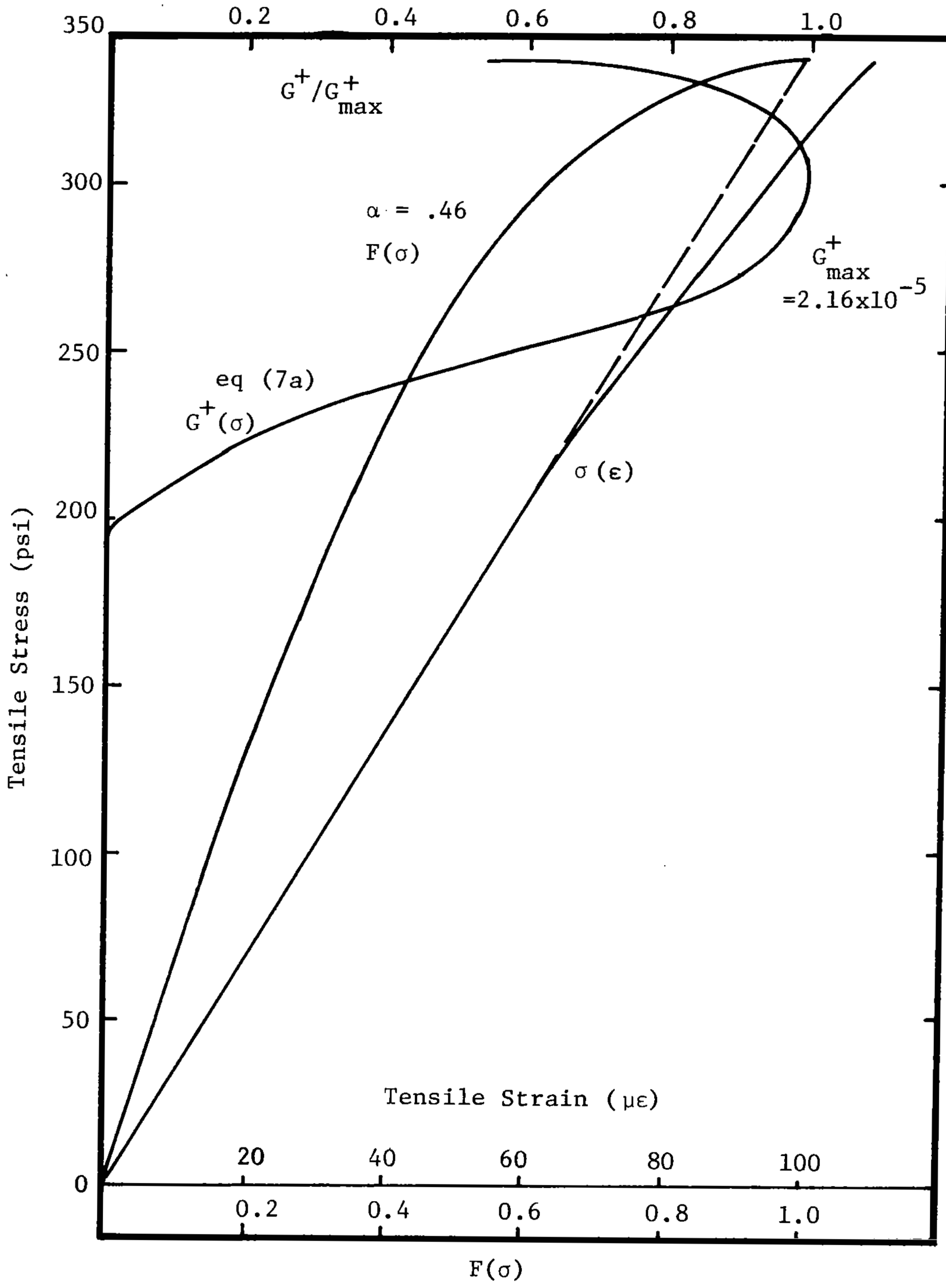

Figure 2.- Data of uniaxial tensile test of concrete.